বিদ্যাসাগর বিশ্ববিদ্যালয়

VIDYASAGAR UNIVERSITY

Question Paper

B.Sc. Honours Examinations 2021

(Under CBCS Pattern)

Semester - II

Subject : MATHEMATICS

Paper : C 3-T

Real Analysis

Full Marks : 60

Time : 3 Hours

Candidates are required to give their answers in their own words as far as practicable.
The figures in the margin indicate full marks.

Answer any ***four*** of the following questions : $4 \times 15 = 60$

1. (a) Define enumerable set. Prove that the set of all sequences whose elements are 0 & 1 is non-enumerable.

   (b) Show that an open interval (a, b) is equivalent to another open interval (c, d).

   (c) Let A and B be two non-empty bounded sets of real numbers. Prove that–

      (i) If $C = \{xy : x \in A, y \in B, x > 0, y > 0\}$

      then Sup C = Sup A.Sup B

(ii) If D = $\{a - b : a \in A, b \in B\}$ then

Sup D = Sup A – Inf B and Inf D = Inf A – Sup B. 5 + 3 + 7

2. (a) Prove that a non-empty bounded closed set is either a singleton or a closed interval or can be obtained from a closed interval by removing a countable number of mutually disjoint open intervals.

(b) Let S = $\{x \in R : x^6 - x^5 \leq 100\}$ and T = $\{x^2 - 2x : x \in (0, \infty)\}$. Prove that the set $S \cap T$ is closed and bounded in R.

(c) Prove that every interior point of an infinite set is an accumulation point. Is the converse true. Justify your answer. 6 + 4 + 5

3. (a) Prove that the necessary and sufficient condition that x_0 be an accumulation point of a set E is that there exist a sequence $\{x_n\}$ of distinct real numbers such that $\lim_{n \to \infty} x_n = x_0$.

(b) Examine whether the following sets are open :

(i) S1 = $\{x \in R : 3x^2 - 10x + 7 > 0\}$

(ii) S2 = $\{x \in R : \cos x \neq 0\}$

(c) Define compact set. Show that $\left\{\frac{x^2}{1+x^2} : x \in R\right\}$ is compact in R. 5 + 5 + 5

4. (a) If $\{x_n\}$ be a sequence such that $x\_n = 2^{2n}\left[1-\cos\left(\frac{1}{2^n}\right)\right]$ then find $\lim_{n \to \infty} x_n$.

(b) Let $\{x_n\}$ be a sequence such that x_1 = a and $x_{n+1} = 1 + \log\left\{\frac{x_n\left(x_n^2+3\right)}{3x_n^2+1}\right\}$

where a ≥ 1.

Show that $\{x_n\}_n$ is convergent. Find the limit.

(c) If $\{a_n\}_n$ converges to '0' and $\{b_n\}$ is bounded then prove that $\lim_{n\to\infty}(a_n b_n) = 0$.

(d) If $\lim_{n\to\infty} x_n = u$ and $\lim_{n\to\infty} y_n = w$ and if $u < w$ prove that there exist m $\in N$ *s.t.* $x_n < y_n$ for all $n > m$. 4 + 5 + 2 + 4

5. (a) Let $\{a_n\}_n$ and $\{b_n\}_n$ be two convergent sequences where $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, then prove that—

(i) $\lim_{n\to\infty}\sqrt{a\_n} = \sqrt{a}$ provided $a \geq 0$ & $a_n \geq 0$ $\forall n \in N$.

(ii) $\lim_{n\to\infty}\frac{a\_n}{b_n} = \frac{a}{b}$ provided $b_n \neq 0$ for all $n \in N$ and $b \neq 0$.

(b) If $\{x_n\}$ be a sequence of real numbers such that $\lim_{n\to\infty}(x_{n+1} - x_n) = c$ where c is a positive real number, then prove that the sequence $\left\{\frac{x_n}{n}\right\}$ converges to c.

(c) If p > 0 and a is a fixed real number, show that $\lim_{n\to\infty}\frac{n^a}{(1+p)^n} = 0$.

6 + 5 + 4

6. (a) Find $\overline{\lim}\ u_n$ and $\underline{\lim}\ u_n$ where u_n =

(i) $(-1)^n\left(1+\frac{1}{n}\right)$ (ii) $\left(\cos\frac{n\pi}{4}\right)^{(-1)^n}$

(b) If $\{u_n\}$ be a cauchy sequence in R having a sub-sequence converging to a real number I, prove that $\lim_{n\to\infty} u_n = I$.

(c) Let $0 < a \le 1$, $x_1 = \frac{a}{2}$ and $x_{n+1} = \frac{1}{2}\left(x_n^2 + a\right) \forall n \in N$. Show that the sequence $\{x_n\}$ is convergent and find its limit.

(d) Prove that $\lim_{n \to \infty} \frac{1}{n}\left\{(2n+1)(2n+2)....(2n+n)\right\}^{\frac{1}{n}} = \frac{27}{4e}$. 4 + 4 + 4 + 3

7. (a) If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series where $a_n = \frac{(-1)^n . n}{2^n}$ and $b_n = \frac{(-1)^n}{\log(n+1)} \forall n \in N$. Prove that $\sum a_n$ is absolutely convergent but $\sum b_n$ is conditionally convergent.

(b) If $\sum u_n$ be a convergent series of positive real numbers, prove that $\sum u_{2n}$ is convergent.

(c) Test the series $\sum u_n$ for convergence where $u_n = \frac{1}{n \log n (\log \log n)}$.

6 + 4 + 5

8. (a) Test the convergence of the following series :

(i) $\frac{1}{\log 2} + \frac{1}{\log 3} + \frac{1}{\log 4} + ...$

(ii) $\frac{1}{4} + \left(\frac{1}{4}\right)^{1+\frac{1}{3}} + \left(\frac{1}{4}\right)^{1+\frac{1}{3}+\frac{1}{5}} + ...$

(b) Test for convergence the series $1 + \frac{2^2}{3^2} + \frac{2^2 \cdot 4^2}{3^2 \cdot 5^2} + \frac{2^2 \cdot 4^2 \cdot 6^2}{3^2 \cdot 5^2 \cdot 7^2} + ...$

(c) Test the convergence of the series $\sum a_n$ where

a_n = $\{2^{-n-\sqrt{n}}$, if n is odd

$\{2^{-n+\sqrt{n}}$, if n is even. 6 + 4 + 5