বিদ্যাসাগর বিশ্ববিদ্যালয়

VIDYASAGAR UNIVERSITY

B.Sc. General Examination 2021

(CBCS)

4th Semester

MATHEMATICS

PAPER—DSC1DT / DSC2DT / DSC3DT

ALGEBRA

Full Marks : 60

Time : 3 Hours

The figures in the right-hand margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

THEORY : DSC1DT

Answer any *four* questions. 4×15

1. (a) State and prove Lagrange's Theorem for a finite group $(G, \bullet)$.

(b) If $(G, \bullet)$ be a group such that $(a \bullet b)^n = a^n \bullet b^n$, for $n = p; p + 1; p + 2$, for all $a; b \in G$. Then prove that G is commutative.

(c) Prove that a non empty subset H of a group (G, •) forms a subgroup if and only if $a, b \in H \Rightarrow a \bullet b^{-1} \in H$. 5+4+6

2. (a) Define cyclic group. Define generator of a cyclic group (G, •). Show that every cyclic group (G, •) is commutative.

(b) Find all elements of order 10 in the group $(\mathbb{Z}_{30}, \oplus)$.

(c) Prove that every group (G, •) of prime order is cyclic.

(2+2+3)+5+3

3. (a) Let (G, •) be a a finite cyclic group generated by a. Then order of G is n if and only if order of a is n.

(b) Let (G, •) be a group and H be a subgroup of G. Prove that any two left cosets of G are either identical or disjoint.

(c) Let (G, •) be a group and H be a subgroup of G. Then a relation ρ on G defined by $\rho = \{(a, b) \in G \times G : a^{-1} \bullet b \in H\}$ is an equivalence relation.

5+6+4

4. (a) Show that intersection of two subgroups of a group (G, •) is again a subgroup of G. Illustrate with an example union of two subgroups of a group (G, •) may not be a subgroup of G.

(b) State and prove under what condition union of two subgroups of a group (G, •) is again a subgroup of G.

(c) Let S_n be the set of all permutations on the set $S = \{1, 2, 3, \ldots, n\}$. Show that S_n is a group with respect to 'multiplication of permutations'. Is this group is cyclic? Justify your answer. (3+2)+5+(4+1)

5. (a) Define normal subgroup of a group (G, •). Find a non-trivial normal subgroup of the group $(S_3, \bullet)$, where S_3 be the set of all permutations on the set $S = \{1, 2, 3\}$.

(b) Show that centre Z(G) of a group (G, •) is a normal subgroup of G. Prove that every subgroup of Z(G) is a normal subgroup of G.

(c) Prove that intersection of two normal subgroups of a group (G, •) is a normal subgroup of the group G. (2+4)+(4+2)+3

6. (a) State and prove necessary and sufficient condition for a subgroup H of a group (G, •) to be a normal subgroup of G.

(b) Let H be a normal subgroup of a group (G, •) and L be a subgroup of G such that H ⊂ L ⊂ G. Prove that H is normal in L.

(c) Show by an example if L be a normal subgroup of a group (G, •) and H be a normal subgroup of H, where H ⊂ L ⊂ G, then H is not necessarily normal in G.

(d) Let H be a normal subgroup of a group (G, •). Let S be the set of all distinct cosets of H in G. Show that S is a group under a suitable composition defined by you. Has this group any special name? (1+4)+2+4+(3+1)

7. (a) Define ideal and sub ring of a ring (R, +, •). Show that every ideal is a subring of a ring (R, +, •).

(b) In the ring $(\mathbb{Z}_n, \oplus, \odot)$, show that an element $\bar{m}$ is a unit if and only if the g.c.d(m, n) = 1.

(c) Prove that every field (F, +, •) is an integral domain. Under what circumstance the converse of the above statement is true? (3+3)+5+(3+1)

8. (a) Let n be a positive integer and let S be the set of *n* *n*-th roots of unity. Show that (S, •) is a cyclic group. Find all possible generators. Have these roots which generate the the group (S, •) any special name?

(b) In a ring $(R, +, \bullet)$, prove that $(-a)\bullet(-b) = a\bullet b$, for all $a, b \in R$.

(c) In a ring $(R, +, \bullet)$, if $(a + b)^2 = a^2 + 2\, a\bullet b + b^2$, for all $a, b \in R$, then prove that R is a commutative ring.

(d) Prove that characteristic of an integral domain is either zero or a prime number. (4+1)+3+3+4