বিদ্যাসাগর বিশ্ববিদ্যালয়

# VIDYASAGAR UNIVERSITY

## Question Paper

## B.Sc. Honours Examination 2021

**(Under CBCS Pattern)**

**Semester - VI**

**Subject: MATHEMATICS**

**Paper: C 14-T**

**(Ring Theory and Linear Algebra II)**

**Full Marks : 60**

**Time : 3 Hours**

*Candiates are required to give their answer in their own words as far as practicable.*

*The figures in the margin indicate full marks.*



Answer any ***four*** questions from the following. 4×15=60

1. (a) Define irreducible element in an integral domain. Prove that every prime element is irreducible in an integral domain. 2+4

   (b) Prove that $x^2+1$ is irreducible over the integer modulo 7. 3

   (c) Find the gcd of the polynomials $f(x)$ and $g(x)$ in the polynomial ring $R[x]$, where $f(x)=[2](x^5-x^4+x^3-x-[1])$, $g(x)=x^4-[2]x^2+[2]$ and $R=\mathbb{Z}_5$. 6

2. (a) Prove that the ring of Gaussian integers $R=\mathbb{Z}+i\mathbb{Z}=\{m+in \mid m, n\in\mathbb{Z}\}$ is a Euclidean domain. 5

(b) Show that an element $x$ in a Euclidean domain is a unit if and only if $d(x) = d(1)$. By using this relation find all units in the ring $\mathbb{Z} + i\mathbb{Z}$ of Gaussian integers. 3+2

(c) (i) Let $R$ be an integral domain with unit element. Then prove that units of $R[x]$ are same as units of $R$.

(ii) Give an example of two polynomials $f(x), g(x) \in R[x]$ such that

$\deg(fg) < \deg(f) + \deg(g)$. 3+2

3. (a) Let $V$ be a finite dimensional vector space over $F$. Let $\hat{V}$ be the dual of $V$ and $\hat{\hat{V}}$ be the double dual of $V$. Define $\psi : V \to \hat{\hat{V}}$ by

$\psi(v) = T_v \ \ \forall v \in V$

where $T_v : \hat{V} \to F$ is such that $T_v(f) = f(v) \ \forall f \in \hat{V}$. Then prove that $\psi$ is an isomorphism from $V$ onto $\hat{\hat{V}}$. 6

(b) If $V$ is a finite dimensional vector space and $v_1 \neq v_2$ are in $V$, prove that there is an $f \in \hat{V}$ such that $f(v_1) \neq f(v_2)$. 4

(c) Let $S = \{\alpha_1, \alpha_2, \alpha_3\}$ be the basis of $R^3$ defined by

$\alpha_1 = (1, 0, -1)$, $\alpha_2 = (1, 1, 1)$, $\alpha_3 = (2, 2, 0)$. Find the dual basis of $S$. 5

4. (a) Define transpose of a linear transformation.

If $T : V \to W$ be a linear transformation and $V$, $W$ are finite dimensionl, then show that

(i) rank of $T$ = rank of $T^t$.

(ii) range of $T^t$ = annihilator of null space of $T$. 2+3+3

(b) If $W$ is a subspace $V$, then define annihilator of $W$, i.e., $A(W)$. Show that $A(W)$ is a subspace of $\hat{V}$. 1+3

(c) Let $V$ be finite dimensional vector space over F and $T \in A(V)$ be an invertible linear transormation whose minimal polynomial is $p(x) = \alpha_0 + \alpha_1 x + ... + x^n$. Prove that $\alpha_0 \neq 0$. 3

5\. (a) Find minimal polynomial of the matrix $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -9 \\ 0 & 1 & 6 \end{pmatrix}$. 5

(b) What is the minimal polynomial of a non-zero.

(i) nilpotent matrix ?

(ii) idempotent matrix ? 3+3

(c) For the matrix $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & -6 \\ 2 & -2 & 3 \end{pmatrix}$, find $P$ such $P^{-1}AP$ is a diagonal matrix. 4

6\. (a) If $A = \begin{pmatrix} -2 & 6 & -6 \\ 0 & 3 & -5 \\ 0 & -3 & 1 \end{pmatrix}$, find eigen values of $A^4 + A^2 + 5A$. 5

(b) Let $V$ be a vector space with basis $\{v_1, v_2, v_3\}$ and let $T : V \to V$ be defined by

$T(v_1) = 3v_1,\ T(v_2) = -v_1 + 2v_2,\ T(v_3) = v_1 - v_2 + 2v_3$.

Then find characteristic polynomial of $T$ and verify Cayley Hamilton Theorem. 6

(c) Let $V$ be a two dimensional vector space over the field $R$ of real numbers. Let $T$ be a linear operator on $V$ such that

$T(v_1) = \alpha v_1 + \beta v_2$,

$T(v_2) = \gamma v_1 + \delta v_2$, $\alpha, \beta, \gamma, \delta \in R$ where $\{v_1, v_2\}$ is a basis of $V$.

Find the condition that 0 be a characteristic root of $T$. 4

7. (a) Define adjoint of a linear transformation $T$.

Prove that adjoint $T^*$ of $T$ satisfies the following properties

(i) $(T_1 + T_2)^* = T_1^* + T_2^*$

(ii) $(T_1 T_2)^* = T_2^* T_1^*$

(iii) $(T^*)^* = T$. 1+2+2+2

(b) If $\alpha$ and $\beta$ be any two vectors in an inner product space $V$, then show that $\|\alpha + \beta\| \le \|\alpha\| + \|\beta\|$. 5

(c) If $N$ is a normal linear transformation and $N(v) = 0$ for $v \in V$, then prove that $N^*(v) = 0$. 3

8. (a) Apply the Gram-Schmidt process to the vectors (1, 0, 1), (1, 0, –1) and (1, 3, 4) to obtain an orthonormal basis for $R^3$ with the standard inner product. 6

(b) Prove that characteristic roots of a Hermitian linear transformation are all real. 4

(c) Reduce the following quadratic form to normal form and then examine whether the quadratic form is positive definite or not.

$6x^2 + y^2 + 18z^2 - 4yz - 12zx$. 5

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