বিদ্যাসাগর বিশ্ববিদ্যালয়

VIDYASAGAR UNIVERSITY

Question Paper

B.Sc. Honours Examination 2021

(Under CBCS Pattern)

Semester - VI

Subject: MATHEMATICS

Paper: C 13-T

(Metric Spaces and Complex Analysis)

Full Marks : 60

Time : 3 Hours

Candiates are required to give their answer in their own words as far as practicable.

The figures in the margin indicate full marks.

Answer any ***four*** questions from the following. 4×15=60

1. (a) Let (X, d) be a metric space and A be a non-empty subset of X. Then prove that a point $x \in A$ is a limit point of the set A in (X, d) if and only if there exists a sequence $\{x_n\}$ of distinct points of $A-\{x\}$ satisfying $\lim_{n\to\infty} x_n = x$.

   (b) Define Cauchy sequence. Prove that a Cauchy sequence in a metric space is convergent if and only if it has a convergent subsequence. 8+7

2. (a) Define complete metric space. Prove that the metric (X, d) is complete, where, X is the space of all bounded sequences $\{\alpha_n\}$ in R and the metric d is defined as

$$d(\{\alpha_n\}, \{\beta_n\}) = \sup\{|\alpha_n - \beta_n| : n \in N\}.$$

(b) Prove that the composition of two continuous functions is continuous in metric space.

8+7

3. (a) Define uniform continuity of a function on a metric space. Let (X, d_x) and (Y, d_y) be two metric spaces and $f:(X, d_x) \to (Y, d_y)$ is a uniformly continuous function. If $\{x_n\}$ is a Cauchy sequence of elements of (X, d_x) then prove that $\{f(x_n)\}$ is a Cauchy sequence of elements of (Y, d_y).

(b) Consider the metric space, (R, d), where R is the set of real numbers and $d(x, y) = |x - y|$. Then prove that the following mapping $f:(R, d) \to (R, d)$ is continuous only at the point $\frac{1}{2}$.

$$f(x) = \begin{cases} x, & x \in Q \\ 1 - x & x \in \bar{Q} \end{cases}$$

where Q is the set of rational numbers. 7+8

4. (a) What do you mean by connected set? Let (X, d) be a metric space and $\{A_i : i \in I\}$ be a family of connected sets such that $\bigcap_{i \in I} A_i \neq \varnothing$, then prove that $\bigcup_{i \in I} A_i$ is connected.

(b) Prove that the continuous image of a compact metric space is compact. 8+7

5. (a) If a series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges to, $f(z)$, at all points interior to some circle $|z - z_0| = R$ then prove that it is the power series expansion for f in powers of $z - z_0$.

(b) Let two complex valued functions $f(z)$ and $g(z)$, $z = x + iy$, be defined on $D \subseteq C$ except possibly at $z_0 = x_0 + iy_0$, such that $\lim_{z \to z_0} f(z) = l_1$ and $\lim_{z \to z_0} g(z) = l_2$, then prove that $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{l_1}{l_2}$, provided $l_2 \neq 0$. 8+7

6. (a) Let $D = \{z : |z| < 1\}$ and the sequence of functions $\{f_n(z)\}$ be defined on *D*, such that, $f_n(z) = z^{n-1}$, $n \in N$, then prove that the series $\sum_{n=1}^{\infty} f_n(z)$ is convergent point-wise on *D* but not uniformly on *D*.

(b) Prove that the function $f(z) = \overline{z}$ is continuous everywhere but not differentiable everywhere.

(c) Suppose a function $f(z)$ be analytic throughout a disk, $|z| < R_0$, then prove that $f(z)$ has the following power series representation.

$$f(z) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} z^n, \ |z| < R_0.$$ 5+4+6

7. (a) Let *f* be analytic inside and on a closed contour *C*, taken in the positive sense and z_0 be a point inside *C*, then prove that

$$\int_C \frac{f(z)}{(z - z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^n(z_0), \ n = 0, 2, 3, \ldots$$

(b) State and prove a sufficient condition of differentiability of a function *f* at a point, z_0 in the complex plane.

(c) Show that the function $u = \cos x + \cosh y$ is harmonic and find its harmonic conjugate. 6+6+3

8. (a) Let $|f(z)| \leq |f(z_0)|$ at each point, *z*, in some neighborhood $|z - z_0| < \varepsilon$, of z_0 in which *f* is analytic, then prove that $f(z)$ has a constant value $f(z_0)$ throughout the neighborhood.

(b) If the function $f(z)$ is analytic and not constant in a given domain *D* then prove that $|f(z)|$ has no maximum value in *D*.