বিদ্যাসাগর বিশ্ববিদ্যালয়

VIDYASAGAR UNIVERSITY

B.Sc. Honours Examination 2021

(CBCS)

4th Semester

MATHEMATICS

PAPER—C9T

MULTIVARIATE CALCULUS

Full Marks : 60

Time : 3 Hours

The figures in the right-hand margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Answer any *four* questions. 4×15

1. (a) Let $f(x,y)=\begin{cases} x\sin\frac{1}{y}+y\sin\frac{1}{x}, xy\neq 0; \\ 0, xy=0. \end{cases}$

Show that at (0,0) the double limit exists but the repeated limits do not exist.

(b) Let $f(x,y)=\begin{cases}\dfrac{2xy}{x^2+y^2}, x^2+y^2\neq 0;\\ 0, x^2+y^2=0.\end{cases}$

Prove that f is a continuous function of either variable when the other variable is given a fixed value. Is f continuous at (0, 0)? Justify.

(c) If $u = f(x,y)$, where $x=r\cos\theta, y=r\sin\theta$; prove that

(i) $\left(\dfrac{\partial u}{\partial x}\right)^2+\left(\dfrac{\partial u}{\partial y}\right)^2=\left(\dfrac{\partial u}{\partial r}\right)^2+\dfrac{1}{r^2}\left(\dfrac{\partial u}{\partial \theta}\right)^2$;

(ii) $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}$. 4+4+7

2. (a) When is a function $f(x,y)$ said to be differentiable at a point (x,y)? State the sufficient condition for differentiability of (x,y).
Verify the sufficient condition for differentiability of the following function

$$f(x,y)=\begin{cases}x^2\sin\dfrac{1}{x}+y^2\sin\dfrac{1}{y}, x\neq 0, y\neq 0;\\ x^2\sin\dfrac{1}{x}, x\neq 0, y=0;\\ y^2\sin\dfrac{1}{y}, x=0, y\neq 0;\\ 0, x=0, y=0.\end{cases}$$

(b) Let $f(x,y)=\begin{cases}\dfrac{x^4+y^4}{x-y}, x\neq y;\\ 0, x=y\ .\end{cases}$

Show that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at (0,0). Examine the continuity of *f(x,y)* at (0, 0).

(c) If H be a homogeneous function in x and y of degree n having continuous first order partial derivatives and $u(x,y)=\left(x^2+y^2\right)^{-n/2}$, show that $\frac{\partial}{\partial x}\left(H\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(H\frac{\partial u}{\partial y}\right)=0$. 6+4+5

3. (a) Let $f(x,y)=\begin{cases}\frac{x^2y}{x^4+y^2},(x,y)\neq(0,0);\\ 0,(x,y)=(0,0).\end{cases}$

Show that f has a directional derivative at (0,0) in any direction $\beta=(l,m),\ l^2+m^2=1$, but f is discontinuous at (0, 0).

(b) If a function *f(x,y)* defined in a certain domain D of the xy-plane where $(a,b)\in D$ be such that both the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist in some neighbourhood of (a,b) and both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are differentiable at *(a,b)*, then prove that $f_{xy}(a,b) = f_{yx}(a,b)$.

(c) For the function $f(x,y)=\begin{cases}\frac{x^2y^2}{x^2+y^2},(x,y)\neq(0,0);\\ 0,(x,y)=(0,0);\end{cases}$

show that $f_{xy}(0,0)=f_{yx}(0,0)$. 4+6+5

4. (a) Prove that the volume of the greatest rectangular parallelepiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ *is* $\frac{8abc}{3\sqrt{3}}$.

(b) Find the stationary points of $f(x,y,z)=x^2y^2z^2$ subject to the condition $x^2+y^2+z^2=a^2$ (x, y, z are positive).

(c) Show that $\iiint (x+y+z)x^2y^2z^2 dx\ dy\ dz = \frac{1}{50400}$ taken throughout the tetrahedron bounded by three coordinate planes and x + y + z = 1.

5+4+6

5. (a) If E be the region bounded by the circle $x^2+y^2-2ax-2by=0$, show that

$$\iint_E \sqrt{x(2a-x)+y(2b-y)}\,dx\ dy = \frac{2\pi}{3}\left(a^2+b^2\right)^{\frac{3}{2}}.$$

(b) Prove that $\iiint_V \dfrac{dx\ dy\ dz}{x^2+y^2+\left(z-\frac{1}{2}\right)^2} = \pi\left(2+\frac{3}{2}\log 3\right)$

where V = $\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2\le 1\}$.

7+8

6. (a) In which direction from the point (1,3,2), the directional derivative of $\phi=2xz-y^2$ is maximum? What is the magnitude of this maximum?

(b) Is there a differentiable vector function $\vec{v}$ such that $curl\ \vec{v}=\vec{r}$? Justify it. Show that $\vec{E}=\frac{\vec{r}}{r^2}$ is irrotational. Find ϕ such that $\vec{E}=-\vec{\nabla}\phi$ and such that $\phi(a)=0$ where a > 0.

(c) If $\vec{A}=\left(4xy-3x^2z^2\right)\hat{i}+2x^2\hat{j}-2x^3z\hat{k}$, prove that $\int_C \vec{A}\bullet d\vec{r}$ is independent of the curve C joining two given points.
Is $\vec{A}\bullet d\vec{r}$ an exact differential? If yes, then solve the differential equation $\vec{A}\bullet d\vec{r}=0$.

3+6+6

7. (a) Prove $\nabla^2 r^n = n(n+1)r^{n-2}$, where n is a constant, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}, r = |\vec{r}|$.

(b) Prove that if $\int_{P_1}^{P_2} \vec{F} \bullet d\vec{r}$ is independent of the path joining any two points P_1 and P_2 in a given region, then $\oint \vec{F} \bullet d\vec{r} = 0$ for all closed paths in the region and conversely.

(c) Verify Green's theorem in the plane for $\oint_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$, where C is the boundary of the region enclosed by : $y = \sqrt{x}, y = x^2$

5+4+6

8. (a) Show that $\vec{F} = (2xy + z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$ is a conservative force field. Find the scalar potential. Also evaluate the work done in moving an object in this field from (1,-2,1) to (3,1,4).

(b) Prove $\iint_S r^5 \hat{n}\, dS = \iiint_V 5r^3 \vec{r}\, dV$, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}, r = |\vec{r}|$.

(c) Evaluate by Stokes' theorem $\oint_C \sin z\, dx - \cos x\, dy + \sin y\, dz$, where C is the boundary of the rectangle : $0 \le x \le \pi, 0 \le y \le 1, z = 3$.

6+5+4

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