বিদ্যাসাগর বিশ্ববিদ্যালয়

VIDYASAGAR UNIVERSITY

B.Sc. Honours Examination 2021

(CBCS)

4th Semester

MATHEMATICS

PAPER—GE4T & GE4P

Full Marks : 60

Time : 3 Hours

The figures in the right-hand margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

GE4T : NUMERICAL METHODS

Answer any *two* questions. 2×15

1. (a) What do you mean by interpolation ? Suppose a table of ten x and y values are given. A polynomial passes through all these point. Can you ensure that it is an interpolating polynomial ? Explain.

(b) Deduce Newton forward interpolation formula with error term. Distinguish it with Newton backward formula. Suppose a table of n values are given. Is the interpolating formula obtained by these formulae same? Explain.

(c) Compare Lagrange and Newton forward interpolation formulae.
(2+2)+(6+2)+3

2. (a) Deduce Newton Raphson method to solve a transcendental equation using Taylor's series. Why it is called method of tangent? Write the merit and demerit of the method. Explain why sometimes bisection method is better than Newton Raphson method.

(b) Find the rate of convergent of the following iteration formulae and conclude which formula is better to find a root of the equation $f(x)=0$.

$$x_{n+1} = x_n - \frac{f(x_n)}{f^n(x_0)} \text{ and } x_{n+1} = x_n - \frac{f(x_n)}{f'(x_0)}.$$ (3+2+2+2)+6

3. (a) Describe 4-point Gauss-Legendra quadrature formula. Is the method applicable for second type improper integral? Among 3-point and 4-point Gauss-Legendra quadrature formulae state with explanation which formula is suitable to find the value of $\int_0^\infty \frac{1}{\sqrt{x}} dx$?

(b) Describe fourth order Runge-Kutta method to solve the first order first degree ordinary differential equation. (5+2+3)+6

4. (a) Describe Gauss-Seidal iteration method to solve a system of n equations and n variables. Write the sufficient condition for convergence and the limitations of this method.

(b) Compare Gauss-Seidal iteration method and matrix inverse method.
(8+2+2)+3

Answer any *one* question. 1×10

5. Describe power method to find the largest (magnitude) eigenvalue and the corresponding eigenvector of a square matrix. Give an outline to find the smallest eigenvalue also. 10

6. Deduce Newton-Cote's quadrature formula and hence Simpson 1/3 formula. 10

PRACTICAL : GE4P

Answer any *one* question. 1×20

1. (a) Write a program to find the sum of the following series

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2148}.$$

(b) Write a program to compute $\int_0^{\pi/2}\sqrt{\cos x}\,dx$ by using Simpson's $\frac{1}{3}$ rule with 50 Sub intervals. 10+10

2. (a) Write a program to find the value of y (0.02) by Euler's modified method from the differential equation $\frac{dy}{dx}=x^2+y, y(0)=1$.

(b) Write a program of fixed point iteration method to solve the equation $\cos x - 3x + 1 = 0$, correct upto 4 decimal places. 10+10

GE4T : PARTIAL DIFFERENTIAL EQUATIONS & APPLICATIONS

Answer any *four* questions. 4×15

1. (a) Eliminate the arbitrary function f from the equation: $f(x^2 + y^2 + z^2, z^2 - 2xy) = 0$ to form the PDE.

(b) Solve the following PDE by Lagrange method : $yzp + zxq = xy$.

6+9

2. (a) Reduce the PDE $p - q = z$ into canonical form and hence find the solution.

(b) Solve the PDE $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ by the method of separation by variables.

8+7

3. (a) Classify the PDE $x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy} = 0$ and reduce it to canonical form.

(b) Determine the solution of wave equation

$u_{tt} = c^2 u_{xx},\ x \in R,\ t > 0$

$u(x,0) = \sin x$

$u_t(x,0) = \cos x$

8+7

4. (a) Derive the one dimensional heat equation.

(b) Solve the non-homogeneous wave equation

$u_{tt} - u_{xx} = t^7,\ x \in R,\ t > 0$

$u(x,0) = 2x + \sin x$

$u_t(x,0) = 0,\ -\infty < x < \infty$.

6+9

5. (a) Find the solution of the equation $u_t = a^2 u_{xx}$ which satisfies the conditions $u(0,t) = u(l,t) = 0, u(x,0) = \frac{x(l-x)}{l^2}$.

(b) State Kepler's law of planetary motion. A particle described an ellipse under a force $\frac{\mu}{(distance)^2}$ towards the focus is projected with a velocity V from a point at distance r from the centre of force. Find the periodic time. 7+8

6. (a) The temperature at one end of a bar 100 cm long with insulated sides is kept at 0°C and other end at 100°C until steady state conditions prevail. The two ends are then suddenly insulated and kept so. Find the temperature distribution.

(b) Find the law of force towards the pole under which the curve $r^n = a^n \cos n\theta$ can be described. 7+8

7. (a) Obtain the differential equation of the central orbit in a plane under the action of a central force F per unit mass.

(b) A particle of mass m moves in a central field of attractive force of which the intensity is $mkr^{-2}e^{-r^2}$ where k is a positive constant. Show that a circular orbit of radius r is stable if $r^2 < \frac{1}{2}$. 7+8

8. (a) A curve is described by a particle having a constant acceleration in a direction inclined at a constant angle to the tangent. Show that the curve is an equiangular spiral.

(b) A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in the equilibrium position. If it is set vibrating by giving to each of its points a velocity $x(1 - x)$, find the displacement of the string at any distance x from one end at any time t. 7+8

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GE4T : RING THEORY AND LINEAR ALGEBRA–I

Answer any *four* questions. 4×15

1. (a) In a ring R, prove that (i) $(-a)(b) = (a)(-b) = -ab$, (ii) $(-a)(-b) = ab$ for all $a, b \in R$.

(b) Define Boolean ring. Show that a Boolean ring is commutative.

(c) Show that a ring R is commutative iff $(a+b)^2 = a^2 + b^2 + 2ab$ for all $a, b \in R$. Show that Z_p modulo p is a field iff p is a prime.

(d) Show that a non-zero element a in Z_n is a unit iff a and n are relatively prime. (2+2)+(1+2)+(2+3)+3

2. (a) Prove that a non-empty subset S of a ring R is a subring of R iff a, $b \in S \Rightarrow ab, a - b \in S$. Find all subrings of the ring $(Z, +, \cdot)$.

(b) What is division ring? If in a ring R, the equation $ax = b$ for all $a, b \in R$ with $(a \neq 0)$ has a solution then show that R is a division ring. If R is a division ring, then show that the centre $Z(R)$ of R is a field.

(c) If D is an integral domain, then prove that characteristic of D is either zero or a prime number. (2+3)+(1+3+3)+3

3. (a) Show that in an integral domain R (with unity) the only idempotents are the zero and unity.

(b) If A is an ideal of a ring R with unity such that $1 \in A$ then show that $A = R$. Determine all the ideals of the ring of integers $(Z, +, \cdot)$.

(c) Prove that a division ring is a simple ring. Let R be a ring with unity, such that R has no right ideals except $\{0\}$ and R. Show that R is a division ring. Show by an example that it is possible to have a ring R with unity where $\{0\}$ and R are the only ideals of R, but R is not a division ring. 3+(2+3)+(2+2+3)

4. (a) Let R be a commutative ring and suppose $px = 0$ for all $x \in R$, where p is a prime number. Then the mapping $f{:}R \to R$ defined by $f(x) = x^p$, $x \in R$, is a homomorphism. Show that $Ker f$ is an ideal of R. Let $f{:}R \to R'$ be an onto homomorphism, where R is a ring with unity. Show that $f(1)$ is unity of R'.

(b) If $f{:}R \to R'$ be an onto homomorphism, then R' is isomorphic to a quotient ring of R. In fact

$R' \cong \frac{R}{Kerf}$. Show that $\frac{\mathbb{Z}}{\langle 2\rangle} = \frac{5\mathbb{Z}}{10\mathbb{Z}}$. (4+2+2)+(4+3)

5. (a) If $(F, +, \cdot)$ be a field, then prove that F is a vector space over F. Show that union of two subspaces may not be a subspace.

Is $W = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_1 - 2x_2 + 3x_3 = 0\}$ a subspace of $\mathbb{R}^3$? Justify.

(b) Define linear span. Prove that $L(S)$ is the smallest subspace of V, containing S.

(c) Show that the vectors $v_1 = (0, 1, -2)$, $v_2 = (1, -1, 1)$, $v_3 = (1, 2, 1)$ are linearly independent in $\mathbb{R}^3(\mathbb{R})$. If $\alpha, \beta, \gamma \in V(F)$ such that $\alpha + \beta + \gamma = 0$, then show that $\{\alpha, \beta\}$ spans the same subspace as $\{\beta, \gamma\}$ i.e., show that $L(\{\alpha, \beta\}) = L(\{\beta, \gamma\})$. (2+2+2)+(1+3)+(2+3)

6. (a) Define basis and dimension of a vector space. If $S = \{v_1, v_2, \ldots, v_n\}$ is a basis of V, then every element of V can be expressed uniquely as a linear combination of $v_1, v_2, \ldots, v_n$. Show that the set $S = \{(l, 2, 1), (2, 1, 0), (1, -1, 2)\}$ forms a basis of $\mathbb{R}^3(\mathbb{R})$.

(b) Show that the set of all real valued continuous functions $y = f(x)$ satisfying the differential equation $y''' + 6y'' + 11y' + 6y = 0$ where

$y' = \frac{dy}{dx}$, is a vector space over $\mathbb{R}$. Also find a basis of this.

(c) Extend the vector (1, 1, 1) to form a basis of $\mathbb{R}^3(\mathbb{R})$. Let $V = M_2(\mathbb{R})$ and let $W = \{A \in V : A = A'\}$ be a subspace of V. Find a basis of W.

(2+3+2)+3+(3+2)

7. (a) Let $V = C\,[a, b]$ and show that T is linear mapping, where $T : V \to \mathbb{R}$ is defined by $T(f) = \int_a^b f(t)dt,\ f \in V$. Let V and W be two vector space over a field F and $T : V \to W$ be a linear mapping then prove that

(i) $T(\theta) = \theta'.\theta,\ \theta'$, are null vectors of V and W respectively.

(ii) T is one-one $\Leftrightarrow Ker\ T = \{\theta\}$.

(b) Let a linear mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$ be degined by $T(x_1, x_2, x_3) = (x_1 + x_2 + x_3,\ 2x_1 + x_2 + 2x_3,\ x_1 + 2x_2 + x_3)$, $(x_1 + x_2 + x_3) \in \mathbb{R}^3$. Find $Ker\,T$, $dim\,N(T)$, $Im(T)$, $dim\,R(T)$.

(c) Let $T : V(F) \to W(F)$ be a linear mapping and V is finite dimensional vector space. Then prove that

$dim\ kerT + dim\ ImT = dim\ V.$ (3+4)+4+4

8. (a) Let $T : V \to W$ be a linear mapping, then prove that $\frac{V}{KerT} \cong R(T)$. Let V and W be vector space over a field F. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a basis of V and $\beta_1, \beta_2, \ldots, \beta_n$ be arbitrary chosen elements (not necessarily distinct) in W. Then prove that there exist one and only one linear mapping $T : V \to W$ such that $T(\alpha_i) = \beta_i$ for $i = 1, 2, \ldots, n$.

(b) A linear mapping $T = \mathbb{R}^3 - \mathbb{R}^2$ is defined by $(x_1, x_2, x_3) = (3x_1 - 2x_2 + x_3), (x_1 - 3x_2 - 2x_3)$, $(x_1, x_2, x_3) \in \mathbb{R}^3$. Find the matrix of T relative to the ordered basis ((1,0,0), (0,1,0), (0,0,1)) of $\mathbb{R}^3$ and ((1,0), (0,1)) of $\mathbb{R}^2$.

(c) The matrix of a linear mapping $T = \mathbb{R}^3 \to \mathbb{R}^2$ relative to the ordered bases ((0,1,0), (1,0,1), (0,1,1)) of $\mathbb{R}^3$ and ((1,0), (1,1)) of $\mathbb{R}^2$ is $\begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 0 \end{pmatrix}$.

Find T. (3+5)+3+4

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GE4T : MULTIVARIATE CALCULUS

Answer any *four* questions. 4×15

1. (a) Identify the points, if any, where the following functions fail to be continuous :

(i) $f(x,y)=\begin{cases} xy \text{ if } xy \geq 0 \\ -xy \text{ if } xy < 0 \end{cases}$ and

(ii) $f(x,y)=\begin{cases} xy \text{ if } xy \text{ is rational} \\ -xy \text{ if } xy \text{ is irrational} \end{cases}$?

(b) Let $f(x,y)=xy\dfrac{x^2-y^2}{x^2+y^2}$ if $(x,y)\neq(0,0)$ and 0 otherwise. Prove that

(i) $f_x\ (0,y) = -y$ and $f_y\ (x,0) = x$ for all x and y.

(ii) f(x,y) is differentiable at (0,0).

(c) Suppose f is a function with $f_x\ (x,y) = f_y\ (x,y)$ for all (x,y). Then show that $f(x,y) = c$, a constant. 4+6+5

2. (a) Find the tangent plane and normal line to $e^{xy^2}+zy^4=61+\dfrac{z^2}{x+1}$ at (0, –2, 8).

(b) Let $f(x,y)=\dfrac{y}{|y|}\sqrt{x^2+y^2}$ if $y\neq 0$ and $f(x,y) = 0$ if $y = 0$. Show that f is continuous at (0,0), it has all directional derivatives at (0,0) but it is not differentiable at (0,0).

(c) Your steel company produces steel boxes at three different plants in amounts x, y and z, respectively, producing an annual revenue of $S(x, y, z) = 8xyz^2 - 200(x + y + z)$. The company is to produce 100 units annually. How should production be distributed to maximize revenue ? 5+5+5

3. (a) Use a triple integral to determine the volume of the region in the first octant that is below $4x + 8y + z = 16$.

(b) Integrate $x^2 - xy + y^2$ over the region $x^2 - xy + y^2 \leq 2$.

(c) Evaluate the integral $\iint_R xy(x+y)\,dxdy$ over the area between the curves $y = x^2$ and $x - y = 0$. 5+5+5

4. (a) Evaluate $\int_0^\infty\int_0^\infty e^{-x^{2(1+y^2)}}x\,dxdy$.

(b) Show that $\int_R r^2 \sin\theta\, dr d\theta = \frac{2a^3}{3}$, where R is the semi-circle $r = 2a\cos\theta$ above the initial line.

(c) Evaluate $\int_S 8 - \frac{\left(x^2+y^2\right)^2}{2}dxdy$ where $S = \{(x,y) : x^2 + y^2 \leq 2\}$. 5+5+5

5. (a) If a scalar field $f : R^3 \to R$ has continuous second partial derivatives, then $Curl(Grad\ f) = 0$.

(b) Determine whether the vector field F $\hat{F} = 2x\hat{i} + (xz-2)\hat{j} + xy\hat{k}$ is conservative.

(c) Evaluate the integral $\int_C xy^2\, d\hat{r}$ where C represents the contour y = x^2 from (0,0). 5+5+5

6. (a) An object moves along the line segment from (0, 0, 0) to (3, 6, 10), subject to the force $F = \langle x^2, y^2, z^2 \rangle$. Find the work done.

(b) Show that the following integral is independent of the path $\int_{(1,0)}^{(3,2)} \left[x + 2ydx + (2x - y)dy\right]$.

(c) State and prove the Fundamental Theorem of Line Integrals.

4+5+6

7. (a) Find the volume enclosed between a sphere of radius a centered on the origin, and a circular cone of half angle α with its vertex at the origin.

(b) State the Green's theorem and use it to evaluate the line integral $\int_C (1 + xy^2)dx - x^2ydy$ where C is the curve consisting of arc of parabola $y = x^2$ from the point (-1,1) to (1,1) and the cord joining these two points.

7+8

8. (a) State Stoke's theorem and verify it for the field $F = \left\langle x^2, 2x, z^2 \right\rangle$ on the ellipse

$S = \left\{(x, y, z) : 4x^2 + y^2 \le 4, z = 0\right\}$.

(b) Verify the divergence theorem for $\iint_S x^2 z^2 dS$, where S is the surface of the sphere $x^2 + y^2 + z^2 = 2021^2$. 8+7

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