বিদ্যাসাগর বিশ্ববিদ্যালয়

VIDYASAGAR UNIVERSITY

B.Sc. Honours Examination 2021

(CBCS)

4th Semester

MATHEMATICS

PAPER—C10T

RING THEORY & LINEAR ALGEBRA – I

Full Marks : 60

Time : 3 Hours

The figures in the right-hand margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Answer any *four* questions. 4×15

1. (a) Let S be the set of all twice differentiable real-valued functions on $\mathbb{R}$ having second derivative zero at $x = 0$. Does $(S, +, \cdot)$ form a ring, where $(f+g)(x) = f(x) + g(x)$ and $(f\cdot g)(x) = f(x)g(x)$ for all $x \in \mathbb{R}$ and $f, g \in S$?

(b) Give an example (with reason) of a left ideal of a ring which is not a right ideal.

(c) Find all homomorphisms from the ring $\mathbb{Z}$ onto itself. Prove that $(\mathbb{Z}, +, \cdot)$ is not isomorphic with $(2\mathbb{Z}, +, \cdot)$ as rings.

(d) Let S be the set of all 2×2 real skew-symmetric matrices over $\mathbb{R}$. Prove that S forms a vector space over R and hence find a basis of S.

(e) Find a linear mapping $T:\mathbb{R}^3 \to \mathbb{R}^3$ so that ker $T = \{(x,y,z)\in\mathbb{R}^3 \,|\, x-y-z=0\}$.

2+2+(2+2)+(2+2)+3

2. (a) Let R be a ring, $x \in R$ and $\langle x\rangle$ denote the smallest ideal of R containing x. Prove that

$$\langle x\rangle = \{rx + xs + \sum_{i=1}^{m} s_i x t_i + nx \mid r, s, s_i, t_i \in R;\ m \in \mathbb{N},\ n \in \mathbb{Z}\}$$

(b) Let I and J be two ideals of a ring R. Show that $R/(I\cap J)$ is isomorphic to a subring of $R/I \times R/J$.

(c) Consider the set of vectors $B=\{\beta_1,\beta_2,\beta_3,\ldots,\beta_k\}$ in a vector space V over the field F. Prove that B is linearly dependent if and only if at least one of the vectors of the set B can be expressed as a linear combination of the remaining vectors of B.

(d) Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator that maps the basis vectors α, β, γ to $\alpha + \beta$, $\beta + \gamma$, γ, respectively. Show that T is an isomorphism.

5+3+4+3

3. (a) In a commutative ring with identity, prove that every maximal ideal is prime. If the ring lacks the identity, then show that the above result may not be true.

(b) Define characteristic of a ring. Give an example of a finite ring whose characteristic is 4.

(c) Let V be the vector space of all 2×2 matrices over the field F. Let W_1 be the subspace of all matrices of the form $\begin{pmatrix} x & -x \\ y & z \end{pmatrix}$ and W_2 be the subspace of all matrices of the form $\begin{pmatrix} a & b \\ -a & c \end{pmatrix}$. Find the dimensions of W_1, W_2, $W_1 + W_2$ and $W_1 \cap W_2$.

(d) Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator defined by

$T(a, b, c) = (3a,\ a - b,\ 2a + b + c)$.

Show that T is invertible. For any $(x,y,z)\in\mathbb{R}^3, T^{-1}(x,y,z)$?

(3+1)+(1+2)+4+(2+2)

4. (a) Does there exist an integral domain with 15 elements? Give justification in support of your answer.

(b) Find all ring homomorphism from $(\mathbb{Z}_{12},+,\cdot)$ to $(\mathbb{Z}_{30},+,\cdot)$.

(c) Let us consider the vector space

$$V = \{A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mid a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{C}, a_{11} + a_{22} = 0\}$$

over the field of real numbers with the usual operations of matrix addition and multiplication of a matrix by a scalar. Find a basis for V and find *dim* V.

(d) Let V be the vector space of all polynomials over $\mathbb{R}$ with degree three or less. Consider the differentiation operator D on V. Then compute the following :

(i) First find the matrix representation of D with respect to the ordered basis $B = \{1, x, x^2, x^3\}$ of V.

(ii) Then consider another basis $B_1 = \{g_1, g_2, g_3, g_4\}$ of V where $g_i(x) = (x+t)^{i-1}$ for all i=1, 2, 3, 4 and for some $t\in\mathbb{R}$. Find the change of basis matrix with respect to B and B_1.

(iii) Hence find the matrix representation of D with respect to the ordered basis B_1 using the change of basis matrix and the matrix representation of D with respect to the ordered basis B.

3+3+3+(2+2+2)

5. (a) Give example of (i) a ring R and a subring S of R such that 1_R, 1_S both exist but $1_R \neq 1_S$; (ii) a ring R and a subring S of R such that 1_S exists but R does not contain the multiplicative identity.

(b) Let R and S be two rings and $f: R \to S$ be an epimorphism. Show that if I is an ideal of R then $f(I)$ is an ideal of S. Give an example to show that the result is not true if f is a homomorphism which is not surjective.

(c) Find four vectors $\alpha, \beta, \gamma, \delta \in \mathbb{R}^4$ such that $\{\alpha, \beta, \gamma, \delta\}$ is a linearly dependent subset but any three of them are linearly independent.

(d) Let V be an n dimensional vector space over the field F and W be an m dimensional vector space over the field F. Then prove that the dimension of the vector space $L(V, W) = \{T: V \to W \mid T \textit{ is a linear transformation}\}$ is mn by constructing a basis of $L(V, W)$.

(2+2)+(2+2)+3+4

6. (a) Let R be a commutative ring with identity $1_R \neq 0$. Prove that an ideal M of R is maximal if and only if the quotient ring R/M becomes a field.

(b) Suppose R is an integral domain and there is a ring homomorphism from $\mathbb{Z}$ onto the integral domain R. Then show that either $R \cong \mathbb{Z}$ or $R \cong \mathbb{Z}_p$ for some prime p.

(c) Extend the set $A = \{(1, 1, 1, 1), (1, -1, 1, -1)\}$ to a basis of the vector space $\mathbb{R}^4$ over $\mathbb{R}$.

(d) Prove that any n dimensional vector space over the field F is isomorphic with the space F^n over F.

4+3+4+4

7. (a) Find the field of quotients of the integral domain $\mathbb{Z}[i]$.

(b) For a commutative ring R with identity, do R and $R[x]$ always have the same characteristic? Justify your answer.

(c) Let V be a vector space over the field F and W be a subspace of V. Show that

$\dim(V/W) = \dim V - \dim W$.

(d) Give examples of two linear operators U, T on the vector space $\mathbb{R}^2$ over $\mathbb{R}$ such that $TU = 0$ but $UT \neq 0$ where 0 denotes the zero operator.

4+3+5+3

8. (a) Let R be an integral domain with finite number of ideals. Show that R is a field. Hence conclude that finite integral domain is a field.

(b) Consider a ring homomorphism $f: R \to S$ where R, S both are commutative rings with identity. If J is a prime ideal of S then is $f^{-1}(J)$ a prime ideal in R? Justify your answer.

(c) Let V be the vector space $\mathbb{R}^4$ over the $\mathbb{R}$ and W be the subspace of V generated by $\{(1, 0, 0, 0), (1, 1, 0, 0)\}$. Find a basis of the quotient space V/W.

(d) Prove that for any two linear operators $T: \mathbb{R}^3 \to \mathbb{R}^2$, $S: \mathbb{R}^2 \to \mathbb{R}^3$, ST is neither one-one nor onto.

(e) Let U, V, W be three finite dimensional vector spaces over a field F. Let $T: V \to W$ be a linear transformation and $S: W \to U$ be an isomorphism. Prove that

$\dim \ker(T) = \dim \ker(ST)$. (3+2)+3+3+2+2

———