বিদ্যাসাগর বিশ্ববিদ্যালয়

VIDYASAGAR UNIVERSITY

B.Sc. Honours Examination 2021

(CBCS)

4th Semester

MATHEMATICS

PAPER—C8T

RIEMANN INTEGRATION AND SERIES OF FUNCTIONS

Full Marks : 60

Time : 3 Hours

The figures in the right-hand margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Answer any *four* questions. 4×15

1. (a) State and prove the necessary and sufficient condition for integrability of a bounded function.

(b) Show by an example that if $|f(x)|$ is integrable then $f(x)$ may not be integrable. 8+7

2. (a) Prove that every continuous function is integrable.

(b) Applying Second Mean Value Theorem of Bonnet's form show that

$$\left|\int_{x'}^{x''} \frac{\sin x}{x} dx\right| \le \frac{2}{x'} \quad \text{where} \quad 0 \le x' \le x''.$$

(c) Show for $k^2 < 1$, $\frac{\pi}{6} \le \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \le \frac{\pi}{6} \times \frac{1}{\sqrt{1-\frac{1}{4} \times k^2}}$. 5+4+6

3. (a) Verify Second Mean Value Theorem of Weierstrass form for the function $x^2 \cos x$ in the Interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(b) Show that $4 \le \int_1^3 \sqrt{3+x^3}\, dx \le 2\sqrt{30}$.

(c) State and prove the fundamental theorem of integral calculus.

5+3+7

4. (a) Show that $\int_0^\infty \frac{\sin x}{x} dx$ is convergent.

(b) Discuss the convergence of $\int_0^\infty e^{-x} x^{n-1} dx$.

(c) Show that $\int_0^\infty \frac{\cos x}{\sqrt{1+x^3}} dx$ converges absolutely but $\int_0^\infty \frac{\cos x}{\sqrt{1+2x^2}} dx$ diverges.

5+5+5

5. (a) Examine the pointwise convergence of the sequence of function $\{f_n\}$ on R defined by $f_n(x) = x^n$.

(b) If $f:\mathbb{R}\to\mathbb{R}$ is uniformly continuous and $f_n(x)=f\left(x+\frac{1}{n}\right)$. Then prove that $\{f_n\}$ converges uniformly to f on $\mathbb{R}$.

(c) Let for each n≥2, $f_n(x)=\begin{cases} n & if \quad 0<n<\frac{1}{2} \\ 0 & otherwise \end{cases}$

Obtain $\lim\limits_{n\to\infty} f_n(x)$ in [0,1] and verify that $\lim\limits_{n\to\infty}\int\limits_0^\infty f_n(x) \neq \int\limits_0^1 lim_{n\to\infty} f_n(x)\,dx$.

4+4+7

6. (a) Check the uniform convergence of the following series:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n}}{n^2\left(1+x^{2n}\right)}, x\in\mathbb{R}$$

(b) Let $f_n(x)=\frac{nx}{1+n^2x^2}-\frac{(n-1)x}{1+(n-1)^2x^2}, x\in[0,1]$.

Show that $\int\limits_0^1\left(\sum\limits_{n=1}^{\infty} f_n(x)\right)dx=\sum\limits_{n=1}^{\infty}\left(\int\limits_0^1 f_n(x)\,dx\right)$; although the series $\sum\limits_{n=1}^{\infty} f_n(x)$ is not uniformly convergent on [0,1].

(c) For the series $\sum\limits_{n=1}^{\infty} f_n(x)$, where $f_n(x)=n^2xe^{-n^2x^2}-(n-1)^2xe^{-(n-1)^2x^2}, x\in[0,1]$,

show that $\sum\limits_{n=1}^{\infty}\int\limits_0^1 f_n(x)\,dx \neq \int\limits_0^1\left(\sum\limits_{n=1}^{\infty} f_n(x)\right)dx$.

Is the series $\sum_{n=1}^{\infty} f_n(x)$ uniformly convergent on $[0,1]$? 3+6+6

7. (a) Expand $f(x) = x$ in Fourier series in the interval $-\pi \le x \le \pi$.

(b) Prove that the even function $f(x) = |x|$ in $-\pi \le x \le \pi$ has a cosine series in Fourier's form.

Use Dirichlet's conditions of convergence to show that the series converges to $|x|$

throughout $-\pi \le x \le \pi$. 7+8

8. (a) Find the radius of convergence of the power series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ and discuss its convergence at each end of the interval.

(b) Find the series for *log (1+x)* by integration and hence use Abel's theorem to show that

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \log 2$. (3+2)+(7+3)